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# On holomorphic framed vertex operator algebras of rank 24

Ching Hung Lam

*Department of Mathematics, National Cheng Kung University,*
Tainan, Taiwan 701
e-mail: chlam@mail.ncku.edu.tw

## 1 Framed vertex operator algebra

This article is based on an on-going research project with H. Yamauchi and H. Shimakura. We shall discuss our ideas on how to classify holomorphic framed vertex operator algebras of rank 24 using the structure codes and $\mathbb{Z}_2$-orbifold constructions.

First, we shall review the notion of a framed vertex operator algebra.

**Definition 1.1.** A Virasoro vector $e$ is called an *Ising vector* if the subalgebra $\mathrm{Vir}(e) \simeq L(1/2, 0)$. Two Virasoro vectors $u, v \in V$ are called *orthogonal* if $[Y(u, z_1), Y(v, z_2)] = 0$. A decomposition $\omega = e^1 + \cdots + e^n$ of the conformal vector $\omega$ of $V$ is called *orthogonal* if $e^i$ are mutually orthogonal Virasoro vectors.

*Remark* 1.2. It is well-known that $L(1/2, 0)$ is rational, $C_2$-cofinite and has three irreducible modules $L(1/2, 0)$, $L(1/2, 1/2)$ and $L(1/2, 1/16)$. The fusion rules of $L(1/2, 0)$-modules are computed in [DMZ]:

$$\begin{aligned} &L(1/2, 1/2) \boxtimes L(1/2, 1/2) = L(1/2, 0), \quad L(1/2, 1/2) \boxtimes L(1/2, 1/16) = L(1/2, 1/16), \\ &L(1/2, 1/16) \boxtimes L(1/2, 1/16) = L(1/2, 0) \oplus L(1/2, 1/2). \end{aligned} \tag{1.1}$$

**Definition 1.3.** ([DGH, M3]) A simple vertex operator algebra $(V, \omega)$ is called *framed* if there exists a set $\{e^1, \dots, e^n\}$ of Ising vectors of $V$ such that $\omega = e^1 + \cdots + e^n$ is an orthogonal decomposition. The full sub VOA $F$ generated by $e^1, \dots, e^n$ is called an *Ising frame* or simply a *frame* of $V$. By abuse of notation, we sometimes call the set of Ising vectors $\{e^1, \dots, e^n\}$ a *frame*, also.

Give a framed VOA $V$ with a frame $T$, one can associate two binary codes $C$ and $D$ to $V$ and $T$ as follows:

Since $T = L(1/2,0)^{\otimes n}$ is a rational vertex operator algebra, $V$ is a completely reducible $T$-module. That is,

$$V \cong \bigoplus_{h_i \in \{0,\frac{1}{2},\frac{1}{16}\}} m_{h_1,\dots,h_n} L(h_1,\dots,h_n),$$

where the nonnegative integer $m_{h_1,\dots,h_n}$ is the multiplicity of $L(h_1,\dots,h_n)$ in $V$. In particular, all the multiplicities are finite and $m_{h_1,\dots,h_n}$ is at most 1 if all $h_i$ are different from $\frac{1}{16}$.

Let $L = L(1/2,h_1) \otimes \cdots \otimes L(1/2,h_n)$ be an irreducible module for $T$. The $\tau$-word $\tau(L)$ of $L$ is a binary word $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{Z}_2^n$ such that

$$\beta_i = \begin{cases} 0 & \text{if } h_i = 0 \text{ or } 1/2, \\ 1 & \text{if } h_i = 1/16. \end{cases} \tag{1.2}$$

For any $\beta \in \mathbb{Z}_2^n$, define $V^\beta$ as the sum of all irreducible submodules $L$ of $V$ such that $\tau(L) = \beta$. Denote $D := \{\beta \in \mathbb{Z}_2^n \mid V^\beta \neq 0\}$. Then $D$ is an even linear subcode of $\mathbb{Z}_2^n$ and $V = \oplus_{\beta\in D} V^\beta$.

For any $c = (c_1,\dots,c_n) \in \mathbb{Z}_2^n$, denote $V(c) = m_{h_1,\dots,h_n} L(h_1,\dots,h_n)$ where $h_i = \frac{1}{2}$ if $c_i = 1$ and $h_i = 0$ elsewhere. Set

$$C := \{c \in \mathbb{Z}_2^n \mid V(c) \neq 0\}.$$

Then $V^0 = \bigoplus_{c\in C} V(c) \neq 0$.

Summarizing, there exists a pair $(C,D)$ of even linear codes such that $V$ is an $D$-graded extension of a code VOA $V_C$ associated to $C$. We call the pair $(C,D)$ the *structure codes* of a framed VOA $V$ associated with the frame $F$. Since the powers of $z$ in an $L(1/2,0)$-intertwining operator of type $L(1/2,1/2) \times L(1/2,1/2) \to L(1/2,1/16)$ are half-integral, the structure codes $(C,D)$ satisfy $C \subset D^\perp$. Moreover, it is known [DGH, M3] that $V$ is holomorphic if and only if $C = D^\perp$.

*Remark* 1.4. Let $V$ be a framed VOA with the structure codes $(C,D)$, where $C,D \subset \mathbb{Z}_2^n$. For a binary codeword $\beta \in \mathbb{Z}_2^n$, we define

$$\tau_\beta(u) := (-1)^{\langle\alpha,\beta\rangle} u \quad \text{for } u \in V^\alpha. \tag{1.3}$$

Then by the fusion rules, $\tau_\beta$ defines an automorphism on $V$ [M1]. Note that the subgroup $P = \{\tau_\beta \mid \beta \in \mathbb{Z}_2^n\}$ is an elementary abelian 2-group and is isomorphic to $\mathbb{Z}_2^n/D^\perp$. In addition, the fixed point subspace $V^P$ is equal to $V^0$ and all $V^\alpha, \alpha \in D$ are irreducible $V^0$-modules. Similarly, we can define an automorphism on $V^0$ by

$$\sigma_\beta(u) := (-1)^{\langle \alpha, \beta \rangle} u \quad \text{for } u \in V(\alpha),$$

where $V^0 = \oplus_{\alpha \in C} V(\alpha)$. Note that the group $Q = \{\sigma_\beta \mid \beta \in \mathbb{Z}_2^n\} \cong \mathbb{Z}_2^n/C^\perp$ is elementary abelian and $(V^0)^Q = V(0)$.

The following theorem is very important to our argument and is proved in [LY]

**Theorem 1.5.** *Let $V = \oplus_{\alpha \in D} V^\alpha$ be a framed VOA with structure codes $(C, D)$. Then*

1. *For every non-zero $\alpha \in D$, the subcode $C_\alpha$ of $C$ contains a doubly even self-dual subcode w.r.t. $\alpha$.*

2. *$C$ is even, every codeword of $D$ has a weight divisible by 8, and $D \subset C \subset D^\perp$.*

As a corollary, the following theorem is also proved in [LY].

**Theorem 1.6.** *Let $V = \oplus_{\alpha \in D} V^\alpha$ be a framed VOA with structure codes $(C, D)$. Then $V = \oplus_{\alpha \in D} V^\alpha$ is a $D$-graded simple current extension of the code VOA $V^0 = V_C$.*

The following corollaries follow immediately by the standard arguments for simple current extensions.

**Corollary 1.7** ([LY, Y2]). *Let $V = \oplus_{\alpha \in D} V^\alpha$ be a framed VOA with structure codes $(C, D)$. Let $W$ be an irreducible $V^0$-module. Then there exists $\eta \in \mathbb{Z}_2^n$, which is unique modulo $D^\perp$, such that $W$ can be uniquely extended to an irreducible $\tau_\eta$-twisted $V$-module which is given by $V \boxtimes_{V^0} W$ as a $V^0$-module. In particular, every irreducible untwisted $V$-module is $D$-stable.*

**Corollary 1.8** ([L1, Y2]). *Let $V = \oplus_{\alpha \in D} V^\alpha$ be a holomorphic framed VOA with structure codes $(C, D)$. For any $\delta \in \mathbb{Z}_2^n$, denote*

$$D^0 = \{\alpha \in D \mid \langle \alpha, \delta \rangle = 0\} \quad \textit{and} \quad D^1 = \{\alpha \in D \mid \langle \alpha, \delta \rangle \neq 0\}.$$

*Define*

$$V(\tau_\delta) = \begin{cases} (\bigoplus_{\alpha\in D^0} V^\alpha) \oplus (\bigoplus_{\alpha\in D^1} M_{\delta+C} \times_{M_C} V^\alpha) & \text{if } \mathrm{wt}\,\delta \text{ is odd,} \\ (\bigoplus_{\alpha\in D^0} V^\alpha) \oplus (\bigoplus_{\alpha\in D^0} M_{\delta+C} \times_{M_C} V^\alpha) & \text{if } \mathrm{wt}\,\delta \text{ is even.} \end{cases}$$

*Then $V(\tau_\delta)$ is also a holomorphic framed VOA. Moreover, the structure codes of $V(\tau_\delta)$ are given by $(C, D)$ if $\mathrm{wt}\,\delta$ is odd and $(C \cup (\delta + C), D^0)$ if $\mathrm{wt}\,\delta$ is even.*

The construction of $V(\tau_\delta)$ is often referred as to a $\mathbb{Z}_2$-orbifold construction.

# 2 Structure codes for holomorphic framed VOAs

If $V$ is a holomorphic framed VOA with the structure codes $(C, D)$, then $C$ will satisfy the following conditions:

1. The length of $C$ is divisible by 16.
2. $C$ is even, every codeword of $C^\perp$ has a weight divisible by 8, and $C^\perp \subset C$.
3. For any $\alpha \in C^\perp$, the subcode $C_\alpha$ of $C$ contains a doubly even self-dual subcode w.r.t. $\alpha$.

For simplicity, we shall call a code $C$ *$F$-admissible* if it satisfies the above conditions (1)- (3). Indeed, one can construct a holomorphic framed VOA starting from an $F$-admissible code.

**Theorem 2.1** ([LY]). *There exists a holomorphic framed VOA with structure codes $(C, C^\perp)$ if and only if $C$ is $F$-admissible, i.e., $C$ satisfies conditions (1)-(3) above.*

*Remark* 2.2. A linear code $C$ is $F$-admissible if and only if its dual $C^\perp$ satisfies the following three conditions:

(i) the length of $C^\perp$ is divisible by 16,

(ii) $C^\perp$ contains the all-one vector,

(iii) $C^\perp$ is *triply even*, that is, $\mathrm{wt}(\alpha)$ is divisible by 8 for any $\alpha \in C^\perp$.

For, let $D$ satisfy the conditions (i), (ii) and (iii) above. Then for any $\alpha, \beta \in D$, the weight of their intersection $\alpha \cdot \beta$ is divisible by 4 and so $\alpha \cdot D$ is doubly even. Then there exists a doubly even code $E$ containing $\alpha \cdot D$ such that $E$ is self-dual w.r.t. $\alpha$. For any $\delta \in (\alpha \cdot D)^{\perp_\alpha}$, we have $\langle \delta, D\rangle = \langle \delta \cdot \alpha, D\rangle = \langle \delta, \alpha \cdot D\rangle = 0$, showing $E \subset (\alpha \cdot D)^{\perp_\alpha} \subset (D^\perp)_\alpha$. Therefore, $D^\perp$ is $F$-admissible.

Now let us consider some examples of $F$-admissible codes.

## 2.1 $\mathbb{Z}_4$ codes and framed VOAs

Let $Z$ be a self-orthogonal linear code over $\mathbb{Z}_4$. Define

$$A_4(Z) = \frac{1}{2}\{(x_1, \dots, x_n) \in \mathbb{Z}^n | (x_1, \dots, x_n) \in Z \mod 4\}.$$

Then $A_4(Z)$ is an even lattice. It is also well-known that $A_4(Z)$ is unimodular iff $Z$ is self-dual. Note that if $Z = 0$, then $A_4(Z) \cong \sqrt{2}A_1^n$. Note that the lattice VOA $V_{\sqrt{2}A_1}$ is a framed VOA (cf. [DMZ, M2]) and

$$V_{\sqrt{2}A_1} \cong L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2})$$

as an $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$-module. Hence the lattice VOA $V_{A_4(Z)}$ is framed for any $Z$.

The positive definite even unimodular lattices of rank 24 have been classified by Niemeier. There are exactly 24 such lattices and they are characterized by their root systems. The following theorem by Kitazume-Harada shows that all positive definite even unimodular lattices of rank 24 can be constructed by $\mathbb{Z}_4$-codes.

**Theorem 2.3** (Kitazume-Harada). *Let $N$ be the Leech lattice or a Niemeier lattice. Then there exists a self-dual $Z_4$ code $Z$ such that $N \cong A_4(Z)$. In particular, the lattice VOA $V_N$ is framed.*

Now let us study the structure codes for the lattice VOA $V_N$.
Let $Z$ be a self-dual $\mathbb{Z}_4$ code. Denote

$$Z_0 = \{(\alpha_1, \dots, \alpha_n) \in \{0,1\}^n | (2\alpha_1, \dots, 2\alpha_n) \in Z\},$$
$$Z_1 = \{\alpha \in \{0,1\}^n | \alpha \equiv \beta \mod 2 \text{ for some } \beta \in Z\}.$$

Then both $Z_0$ and $Z_1$ are even binary codes. Moreover, $Z_1$ is doubly even and $Z_0^\perp = Z_1$.

**Proposition 2.4.** *Let $Z$ be a self-dual linear $\mathbb{Z}_4$-code and $Z_0$ and $Z_1$ defined as above. Then the structure codes of the lattice VOA $V = V_{A_4(Z)}$ are given by*

$$D = d(Z_1) \quad \textit{and} \quad C = D^\perp,$$

*where $d : \mathbb{Z}_2^n \to \mathbb{Z}_2^{2n}$ is given by $d(a_1, a_2, \dots, a_n) = (a_1, a_1, a_2, a_2, \dots, a_n, a_n)$. Note that the code $\{(0,0), (1,1)\}^n$ is contained in $C$.*

Let $\delta = (10)^n$. Then $\tau_\delta$ defines an automorphism on $V_{A_4(Z)}$. In fact, $\tau_\delta$ is conjugate to the automorphism $\theta$, which is the lift of the $-1$-map on the lattice $N = A_4(Z)$.

By Corollary 1.8, the structure codes for the $\tau_\delta$-twisted orbifold

$$V(\tau_\delta) = (\bigoplus_{\alpha \in D^0} V^\alpha) \oplus (\bigoplus_{\alpha \in D^0} M_{\delta+C} \times_{M_C} V^\alpha)$$

are given by $D =< d(Z_1), (10)^n >$ and $C = D^\perp$. Note that $C$ contains the code $E^+$ generated by

$$\begin{array}{l} 11110000\cdots \\ 00111100\cdots \\ 00001111\cdots \\ 0000001111\cdots \end{array}$$

We believe that this case is the typical case and the following holds.

**Conjecture 1.** Let $D$ be a indecomposable triply even binary code of length $16k$. Then $D$ is a subcode of $< d(C), (01)^{8k} >$, where $C$ is a double even self-dual codes of length $8k$.

A code is said to be *indecomposable* if there does not a partition $I \cup J = \{1, \ldots, n\}$ such that $I \cap J = \emptyset$ but $D = D_1 \oplus D_2$ with $\operatorname{supp} D_1 \subset I$ and $\operatorname{supp} D_2 \subset J$.

The conjecture actually holds for $k = 1, 2$ but is not proved for $k \geq 3$. If the conjecture also holds for $k = 3$, then we have the following classification of holomorphic framed VOAs of rank 24.

**Theorem 2.5.** *Assume that Conjecture 1 holds for* $k = 3$. *If* $V$ *is holomorphic framed VOA of rank 24, then* $V$ *is isomorphic to one of the following:*

*(1) a lattice VOA* $V_N$, *or*

*(2) the* $\theta$*-orbifold of a lattice VOA* $V_N$,

*where* $N$ *is the Leech lattice or a Niemeier lattice.*

*In particular,* $V$ *can be characterized by its weight* 1 *subspace* $V_1$.

*Remark* 2.6. Note that if $N \subset A_1^{24}$, then $\tau_\delta$-orbifold is again a lattice VOA. There are 9 such cases and hence there are exactly 15 holomorphic framed VOAs of rank 24,which are not lattice VOAs. Therefore, there are totally 39 holomorphic framed VOAs of rank 24 – 24 lattice VOAs and 15 $\theta$-twisted orbifolds.

**Sketch of the proof**

Let $(C, D)$ be the structure codes of $V$. If the conjecture holds, then $C \supset E^+$, which is generated by

$$\begin{array}{l} 11110000\cdots \\ 00111100\cdots \\ 00001111\cdots \\ 0000001111\cdots \end{array}$$

If $C$ contains $\{(00),(11)\}^{24}$, then $V$ contains a subVOA isomorphic to $V_{\sqrt{2}A_1}^{\otimes 24}$ and hence $V$ is isomorphic to a lattice VOA associated with a Niemeier lattice.

Otherwise, let $\alpha = (1100\ldots0)$. Then $\alpha \notin C$ and the $\tau_\alpha$-orbifold of $V$ will have the structure code containing $\{(00),(11)\}^{24}$ and hence the $\tau_\delta$-orbifold $V(\tau_\delta)$ is isomorphic to a lattice VOA $V_N$. By reversing the orbifold construction, one can show that $V$ is a $\theta$-twisted orbifold of $V_N$.

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